Extended phase space thermodynamics and $P-V$ criticality of charged black holes in Brans-Dicke theory

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In this paper, taking in to account Brans-Dick theory, we investigate thermodynamic behavior of charged black hole solutions. We study the analogy of the black hole solution with the Van der Waals liquid-gas system in the extended phase space by considering the cosmological constant as dynamical pressure. We obtain critical values of thermodynamic coordinates and plot $P-r_+$ and $G-T$ diagrams to study the phase transition.

I. INTRODUCTION

Einstein constructed the theory of general relativity that describes the dynamics of our solar system well enough, but it probably does not describe gravity accurately at all scales. The problem that general relativity faced is that, it does not accommodate either Mach's principle or Dirac's large-number hypothesis. It is unable to describe the accelerated expansion of the universe accurately. Herein cosmologists explored various alternative theories of gravity [1]. Brans and Dicke were pioneers in studying these alternative theories and they developed another relativistic theory known as Brans-Dicke (BD) theory [2]. This theory can be regarded as an economic modification of Einstein general relativity which describes gravitation in terms of metric as well as a scalar field and it accommodate both Mach's principle and Dirac's large- number hypothesis. Due to the importance of black holes and gravitational collapse in both classical and quantum gravity, authors have investigated various aspects of them in BD theory [3]. It has been proved that in four dimensions, the stationary and vacuum BD solution is just the Kerr solution with a constant scalar field [4]. In order to investigate the distinction between the BD theory and Einstein theory Cai and Myung proved that the black hole solution in the BD–Maxwell theory in four dimensions is just the Reissner–Nordström (RN) solution with a trivial scalar field [5]. In higher dimensions, however, it would be the RN solution with a non-trivial scalar field. This is because the stress energy tensor of Maxwell field is not traceless in higher dimensions and the action of Maxwell field is not invariant under the conformal transition.

On the other hand, thermodynamic properties of the black holes have been fascinating subject for many years. It was found out that black holes along all assigned thermodynamic variables also have rich phase structure in complete analogy with non-gravitational thermodynamic system similar to van der Waals gas system. With the conception of expecting the cosmological constant term to arise from the vacuum expectation value of a quantum field, we can assume that it can vary. Hence, we can treat the cosmological constant and its conjugate as dynamical pressure and volume of a black hole system respectively [6–8]. Studying the thermodynamics of black holes in AdS space time has exhibit various phase transitions with the same critical behavior as van der Waals model, qualitatively [9]. The paper of Hawking was the initial studies on this subject [10]. He pointed out there is a thermal radiation (black hole first order phase transition) for Schwarzschild- AdS black hole space time. Adding charge and/or radiation will result a behavior similar to a van der Waals liquid/gas [11–13] and the analogy will improve by being in the extended phase space where the cosmological constant is interpreted as thermodynamical pressure.

In this paper, we want to investigate the thermodynamic phase transition of charged black holes in BD theory by using the analogy between our system and the van der Waals liquid/gas.

The outline of our paper is as follows. Section II is devoted to brief review of BD–Maxwell field equations with their relation with dilaton gravity by a conformal transformation. In Sec. III, we obtain charged black hole solutions of both dilaton gravity and BD theory. Next, we extend the phase space by considering cosmological constant as thermodynamic pressure and calculate critical values, and then we plot diagrams for different cases in Sec. IV. In next section, we give a detailed discussion regarding diagrams, their physical interpretations, and the effects of BD parameter. We finish our paper with some closing remarks.

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II. FIELD EQUATIONS AND CONFORMAL TRANSFORMATIONS

The action of $(n+1)$- dimensional BD–Maxwell theory with a scalar field Φ and a self-interacting potential $V(\Phi)$ can be written as [5]

$$I_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}\left(\Phi\mathcal{R} \;-\frac{\omega}{\Phi}(\nabla\Phi)^2 - V(\Phi) - F_{\mu\nu}F^{\mu\nu}\right), \tag{1}$$

where $\mathcal{R}$ is the scalar curvature, the factor ω is the coupling constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field and A_μ is the electromagnetic potential. Varying the action (1) with respect to the gravitational field $g_{\mu\nu}$, the scalar field Φ and the gauge field A_μ, one can obtain equations of motion with the following explicit forms [5]

$$\begin{aligned} G_{\mu\nu} &= \frac{\omega}{\Phi^2}\left(\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}(\nabla\Phi)^2\right) - \frac{V(\Phi)}{2\Phi}g_{\mu\nu} + \frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi\right) \\ &\quad + \frac{2}{\Phi}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\right), \end{aligned} \tag{2}$$

$$\nabla^2\Phi = -\frac{n-3}{2\left[(n-1)\,\omega + n\right]}F^2 + \frac{1}{2\left[(n-1)\,\omega + n)\right]}\left[(n-1)\Phi\frac{dV(\Phi)}{d\Phi} - (n+1)\,V(\Phi)\right], \tag{3}$$

$$\nabla_\mu F^{\mu\nu} = 0, \tag{4}$$

where $G_{\mu\nu}$ and ∇_μ are, respectively, the Einstein tensor and covariant derivative of manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. Due to the appearance of the second derivatives of scalar field in the right hand side of (2), solving the field equations (2)-(4) directly is a non-trivial task. Using a suitable conformal transformation, one can remove this difficulty. Indeed, via the conformal transformation [5] the BD–Maxwell theory can be transformed into the Einstein- Maxwell theory with a minimally coupled scalar dilaton field. Suitable conformal transformation can be shown as

$$\begin{aligned} \bar{g}_{\mu\nu} &= \Phi^{2/(n-1)}g_{\mu\nu}, \\ \bar{\Phi} &= \frac{n-3}{4\alpha}\ln\Phi, \end{aligned} \tag{5}$$

where

$$\alpha = (n-3)/\sqrt{4(n-1)\omega + 4n}. \tag{6}$$

It is notable that all functions and quantities in Jordan frame ($g_{\mu\nu}$, Φ and $F_{\mu\nu}$) can be transformed into Einstein frame ($\bar{g}_{\mu\nu}$, $\bar{\Phi}$ and $\bar{F}_{\mu\nu}$). Applying the mentioned conformal transformation on the BD action (1), one can obtain action of dilaton gravity

$$\bar{I}_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^{n+1}x\sqrt{-\bar{g}}\left\{\bar{\mathcal{R}} - \frac{4}{n-1}(\bar{\nabla}\bar{\Phi})^2 - \bar{V}(\bar{\Phi}) - \exp\left(-\frac{4\alpha\bar{\Phi}}{(n-1)}\right)\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}\right\}, \tag{7}$$

where $\bar{\mathcal{R}}$ and $\bar{\nabla}$ are, respectively, the Ricci scalar and covariant derivative corresponding to the metric $\bar{g}_{\mu\nu}$, and $\bar{V}(\bar{\Phi})$ is

$$\bar{V}(\bar{\Phi}) = \Phi^{-(n+1)/(n-1)}V(\Phi). \tag{8}$$

Regarding $(n+1)-$dimensional Einstein–Maxwell–dilaton action (7), α is an arbitrary constant that governs the strength between the dilaton and Maxwell fields. One can obtain the equations of motion by varying this action (7) with respect to $\bar{g}_{\mu\nu}$, $\bar{\Phi}$ and $\bar{F}_{\mu\nu}$

$$\bar{\mathcal{R}}_{\mu\nu} = \frac{4}{n-1}\left(\bar{\nabla}_\mu\bar{\Phi}\bar{\nabla}_\nu\bar{\Phi} + \frac{1}{4}\bar{V}\bar{g}_{\mu\nu}\right) + 2e^{-4\alpha\bar{\Phi}/(n-1)}\left(\bar{F}_{\mu\lambda}\bar{F}_\nu{}^\lambda - \frac{1}{2(n-1)}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}\bar{g}_{\mu\nu}\right), \tag{9}$$

$$\bar{\nabla}^2\bar{\Phi} = \frac{n-1}{8}\frac{\partial\bar{V}}{\partial\bar{\Phi}} - \frac{\alpha}{2}e^{-4\alpha\bar{\Phi}/(n-1)}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}, \tag{10}$$

$$\partial_\mu\left[\sqrt{-\bar{g}}e^{-4\alpha\bar{\Phi}/(n-1)}\bar{F}^{\mu\nu}\right] = 0 \tag{11}$$

By assuming the $\left(\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \bar{\Phi}\right)$ as solutions of Eqs. (9)-(11) with potential $\bar{V}\left(\bar{\Phi}\right)$ and comparing Eqs. (2)-(4) with Eqs. (9)-(11) we find the solutions of Eqs. (2)-(4) with potential $V(\Phi)$ can be written as

$$[g_{\mu\nu}, F_{\mu\nu}, \Phi] = \left[\exp\left(-\frac{8\alpha\bar{\Phi}}{(n-1)\,(n-3)}\right)\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \exp\left(\frac{4\alpha\bar{\Phi}}{n-3}\right)\right]. \tag{12}$$

III. CHARGED SOLUTIONS IN $(n+1)-$ DIMENSIONS

Our strategy is to construct the solutions of BD theory with $n \geqslant 4$ and the quadratic potential

$$V(\Phi) = 2\Lambda\Phi^2. \tag{13}$$

Applying the conformal transformation (5), the potential $\bar{V}(\bar{\Phi})$ becomes a Liouville-type potential

$$\bar{V}(\bar{\Phi}) = 2\Lambda \exp\left(\frac{4\alpha\bar{\Phi}}{n-1}\right). \tag{14}$$

In other words, instead of solving Eqs. (2)-(4) with quadratic potential, we solve Eqs. (9)-(11) with Liouville-type potential. Assuming the $(n+1)-$dimensional metric

$$d\bar{s}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2R^2(r)d\Omega_{n-1}^2, \tag{15}$$

where $d\Omega_{n-1}^2$ is the metric of a unit $(n-1)-$sphere, and $f(r)$ and $R(r)$ are metric functions. By integrating the Maxwell equation (11), we can obtain the nonzero electric field $\bar{F}_{tr}$ as

$$\bar{F}_{tr} = \frac{q}{(rR)^{n-1}} \exp\left(\frac{4\alpha\bar{\Phi}}{n-1}\right). \tag{16}$$

Taking into account the metric (15) with Maxwell field (16), the solutions of (9) and (10) are

$$\begin{aligned} f(r) &= -\frac{(n-2)\left(\alpha^2+1\right)^2 c^{-2\gamma} r^{2\gamma}}{(\alpha^2+n-2)(\alpha^2-1)} + \frac{2\Lambda(\alpha^2+1)^2c^{2\gamma}}{(n-1)(\alpha^2-n)}r^{2(1-\gamma)} - \frac{m}{r^{(n-2)}}r^{(n-1)\gamma} \\ &\quad + \frac{2q^2(\alpha^2+1)^2c^{-2(n-2)\gamma}}{(n-1)(\alpha^2+n-2)r^{2(n-2)(1-\gamma)}}, \end{aligned} \tag{17}$$

$$R(r) = \exp\left(\frac{2\alpha\bar{\Phi}}{n-1}\right) = \left(\frac{c}{r}\right)^{\gamma}, \tag{18}$$

$$\bar{\Phi}(r) = \frac{(n-1)\alpha}{2(1+\alpha^2)} \ln\left(\frac{c}{r}\right), \tag{19}$$

where m is an integration constant which is related to the total mass, c is another arbitrary constant related to the scalar field and $\gamma = \alpha^2/(1+\alpha^2)$.

Now, we are in a position to obtain the solutions of Eqs. (2)-(4) by using the conformal transformation. Considering the following spherically symmetric metric

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{V(r)} + r^2H^2(r)d\Omega_{n-1}^2, \tag{20}$$

with Eqs. (2)-(4), we find that the functions $U(r)$ and $V(r)$ are

$$\begin{aligned} U(r) &= \frac{2\Lambda(\alpha^2+1)^2c^{2\gamma\left(\frac{n-5}{n-3}\right)}}{(n-1)(\alpha^2-n)}r^{2\left(1-\frac{\gamma(n-5)}{n-3}\right)} - \frac{mc^{\left(\frac{-4\gamma}{n-3}\right)}}{r^{(n-2)}}r^{\gamma\left(n-1+\frac{4}{n-3}\right)} \\ &\quad + \frac{2q^2(\alpha^2+1)^2c^{-2\gamma\left(n-2+\frac{2}{n-3}\right)}}{(n-1)(\alpha^2+n-2)r^{2\left[(n-2)(1-\gamma)-\frac{2\gamma}{n-3}\right]}} - \frac{(n-2)\left(\alpha^2+1\right)^2}{(\alpha^2+n-2)(\alpha^2-1)}\left(\frac{c}{r}\right)^{-2\gamma\left(\frac{n-1}{n-3}\right)}, \end{aligned} \tag{21}$$

$$\begin{aligned} V(r) &= \frac{2\Lambda(\alpha^2+1)^2c^{2\gamma\left(\frac{n-1}{n-3}\right)}}{(n-1)(\alpha^2-n)}r^{2\left(1-\frac{\gamma(n-1)}{n-3}\right)} - \frac{mc^{\left(\frac{4\gamma}{n-3}\right)}}{r^{(n-2)}}r^{\gamma\left(n-1-\frac{4}{n-3}\right)} \\ &\quad + \frac{2q^2(\alpha^2+1)^2c^{-2\gamma\left(n-2-\frac{2}{n-3}\right)}}{(n-1)(\alpha^2+n-2)r^{2\left[(n-2)(1-\gamma)+\frac{2\gamma}{n-3}\right]}} - \frac{2(n-2)\left(\alpha^2+1\right)^2}{(\alpha^2+n-2)(\alpha^2-1)}\left(\frac{c}{r}\right)^{-2\gamma\left(\frac{n-5}{n-3}\right)}. \end{aligned} \tag{22}$$

Using the conformal transformation the electromagnetic field becomes

$$F_{tr} = \frac{qc^{(3-n)\gamma}}{r^{(n-3)(1-\gamma)+2}}. \tag{23}$$

As one can see electromagnetic field becomes zero as $r \longrightarrow \infty$. It is also notable that obtained solutions are just the charged solutions of Einstein gravity (Reissner- Nordström AdS black hole) as $\omega \longrightarrow \infty$.

IV. EXTENDED PHASE SPACE AND P- V CRITICALITY IN BD BLACK HOLES

Calculations show that the Hawking temperature of a BD black hole on the outer horizon r_+ is

$$T=\frac{\kappa}{2\pi}=\frac{1}{4\pi}\sqrt{\frac{V}{U}}\left(\frac{dU}{dr}\right)\bigg|_{r=r_+}, \tag{24}$$

where κ is the surface gravity. After some simplifications, we obtain

$$\begin{aligned} T &= -\frac{2(1+\alpha^2)}{4\pi(n-1)}\left(\Lambda b^{2\gamma} r_+^{1-2\gamma}+\frac{q^2 b^{-2(n-2)\gamma}}{{r_+}^{\gamma}}{r_+}^{(2n-3)(\gamma-1)}\right) \\ &\quad +\frac{\left[\gamma\left(n-3\right)-n+2\right]\left(\alpha^2+1\right)^2\left(n-2\right)}{4\pi r_+\left(\alpha^2+n-2\right)\left(\alpha^2-1\right)}\left(\frac{c}{r_+}\right)^{-2\gamma}, \end{aligned} \tag{25}$$

which is invariant under the conformal transformation because the conformal parameter is regular at the horizon. The finite mass and the entropy of the black hole can be obtained by using the Euclidian action [5]

$$M = \frac{c^{(n-1)\gamma}}{16\pi}\left(\frac{n-1}{1+\alpha^2}\right)m, \tag{26}$$

$$S = \frac{c^{(n-1)\gamma}}{4}r_+^{(n-1)(1-\gamma)}. \tag{27}$$

We regard Λ and its corresponding conjugate quantity as the thermodynamic pressure $P=\frac{-\Lambda}{8\pi}$ and volume respectively

$$V=\left(\frac{\partial H}{\partial P}\right)_{S,Q}=\left(\frac{\partial M}{\partial P}\right)_{S,Q}. \tag{28}$$

Herein, we are interested in studying the phase transition of this black hole. The equation of state of the black hole can be obtain using equation (25)

$$\begin{aligned} P &= \frac{q^2\left(n-1\right)}{8\pi r_+^{(n-1)}}\left(\frac{b}{r_+}\right)^{-2\gamma(n-1)}+\frac{T\left(n-1\right)}{4\left(\alpha^2+1\right)r_+}\left(\frac{b}{r_+}\right)^{-2\gamma} \\ &\quad -\frac{\left(n-1\right)\left(n-2\right)\left(\alpha^2+1\right)\left[\gamma\left(n-3\right)-n+2\right]}{16\pi\left(\alpha^2-1\right)\left(\alpha^2+n-2\right)}r_+^{2(\gamma-1)}\left(\frac{bc}{r_+}\right)^{-2\gamma}, \end{aligned} \tag{29}$$

where r_+ is linear function of the specific volume v in geometric unit [9].

We can investigate the existence of phase transition and critical behavior of this black hole by plotting and analyzing the graphs of $P-v$ and $G-T$ diagrams. One may use the inflection point properties

$$\left(\frac{\partial P}{\partial v}\right)_T=0,$$
$$\left(\frac{\partial^2 P}{\partial v^2}\right)_T=0,$$

to obtain the critical values for the temperature, pressure and volume. Due to the difficulties of solving these equations analytically, we use the numerical method to obtain critical values.

According to first law of black hole thermodynamics and the interpretation of M (total mass of black hole) as H (the black hole enthalpy) [14] the Gibbs free energy of black hole can be written as

$$G=H-TS=M-TS \tag{30}$$

$$\begin{aligned} G &= \frac{\left(\alpha^2+1\right)\left(n-1\right)\left(n-2\right)r_+^{n-2}}{16\pi\left(\alpha^2+n-2\right)\left(1-\alpha^2\right)}\left(\frac{c}{r_+}\right)^{\gamma(n-3)}\left(1+\frac{\left(\alpha^2+1\right)\left[\left(\gamma-1\right)n-3\gamma+2\right]}{n-1}\right) \\ &\quad -\frac{P\left(\alpha^2+1\right)}{r_+^{\gamma(n+1)-n}}\left(\frac{b^{2\gamma}c^{\gamma(n-1)}}{n-1}+\frac{c^{\gamma(n+1)}}{\alpha^2-n}\right) \\ &\quad -\frac{q^2 r_+^{\gamma(n-3)-n+2}}{\pi}\left(\frac{c^{\gamma(n-1)}b^{-2\gamma(n-2)}}{(n-1)}+\frac{c^{-\gamma(n-3)}}{(\alpha^2+n-2)}\right). \end{aligned} \tag{31}$$

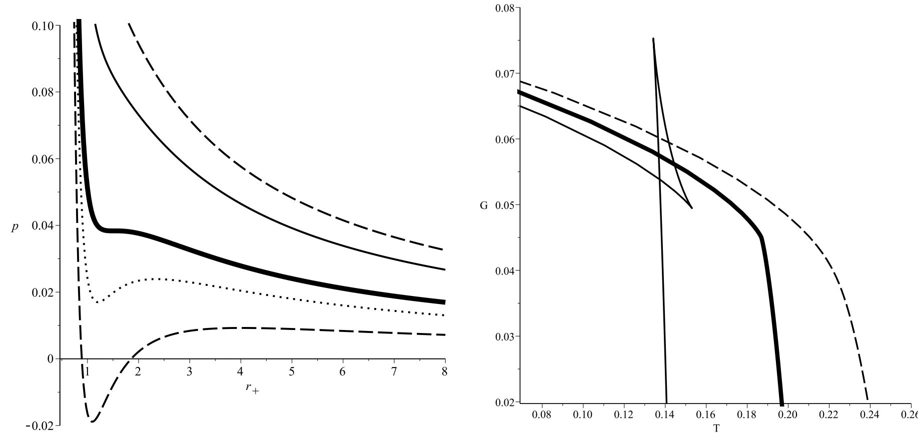

FIG. 1: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 4$, $q = 1$ and $\omega = 1$.
$P - r_+$ diagram, from up to bottom $T = 1.8T_c$, $T = 1.5T_c$, $T = T_c$, $T = 0.8T_c$ and $T = 0.5T_c$ respectively.
$G - T$ diagram, from up to bottom $P = 1.5P_c$, $P = P_c$ and $P = 0.5P_c$, respectively.

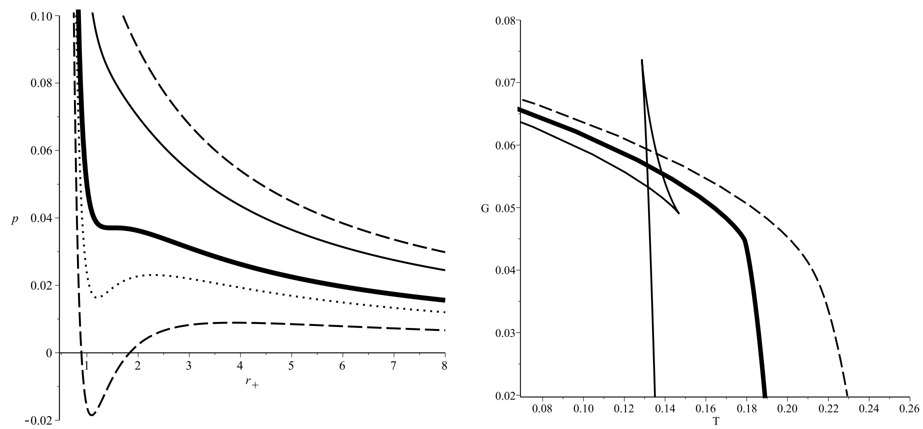

FIG. 2: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 4$, $q = 1$ and $\omega = 3$.
$P - r_+$ diagram, from up to bottom $T = 1.8T_c$, $T = 1.5T_c$, $T = T_c$, $T = 0.8T_c$ and $T = 0.5T_c$ respectively.
$G - T$ diagram, from up to bottom $P = 1.5P_c$, $P = P_c$ and $P = 0.5P_c$, respectively.

The behavior of Gibbs free energy with respect to temperature may be investigated by plotting the graph of $G - T$. We will see the characteristics swallow-tail behavior which guarantees the existence of the phase transitions.

V. DISCUSSION ON THE RESULTS OF DIAGRAMS

Thermodynamical behavior of the system is shown in Figs. 1-9. These critical values represent phase transition points, which one can see in $G - T$ and $P - r_+$ diagrams. Studying $P - r_+$ graphs (left panel of Figs. 1-9) show that obtained values are critical points in which phase transition occurs. It can be seen in the graphs that as the coupling constant (ω) increases, the temperature in which phase transition occurs decreases. Increasing ω leads to decreasing in the value of Gibbs free energy of phase transition point (see Fig. 7 for more details). The results indicate that with larger coupling constant the energy that system needs in order to have phase transition becomes less. It is also evident from the effect of the coupling constant on the total finite mass of the black hole that by increasing the coupling constant the total finite mass which is interpreted as enthalpy of the system will increase too. It means that in order to have phase transition it is expected that the mentioned black hole absorb more mass from surrounding.

Studying $P - r_+$ diagrams shows that as coupling constant increases, both critical pressure and horizon radius of critical point decrease. On the other hand, it is worthwhile to mention that due to the relation between pressure and cosmological constant (which is related to the asymptotical curvature of the background), as the coupling constant increases, the necessity of having a background with more curvature increases too.

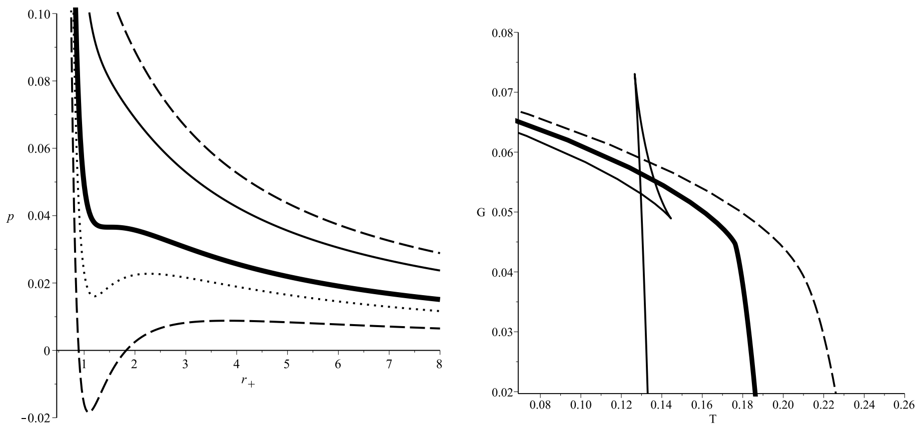

FIG. 3: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 4$, $q = 1$ and $\omega = 5$.
$P - r_+$ diagram, from up to bottom $T = 1.8T_c$, $T = 1.5T_c$, $T = T_c$, $T = 0.8T_c$ and $T = 0.5T_c$ respectively.
$G - T$ diagram, from up to bottom $P = 1.5P_c$, $P = P_c$ and $P = 0.5P_c$, respectively.

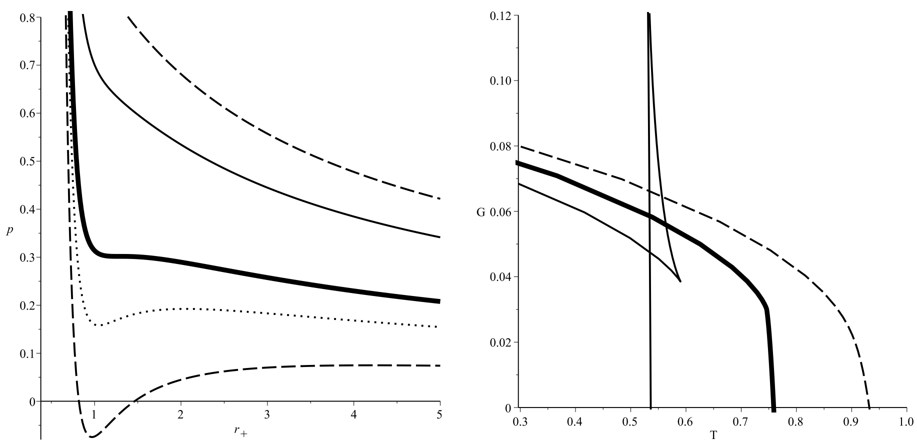

FIG. 4: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 6$, $q = 1$ and $\omega = 1$.
$P - r_+$ diagram, from up to bottom $T = 1.8T_c$, $T = 1.5T_c$, $T = T_c$, $T = 0.8T_c$ and $T = 0.5T_c$ respectively.
$G - T$ diagram, from up to bottom $P = 1.5P_c$, $P = P_c$ and $P = 0.5P_c$, respectively.

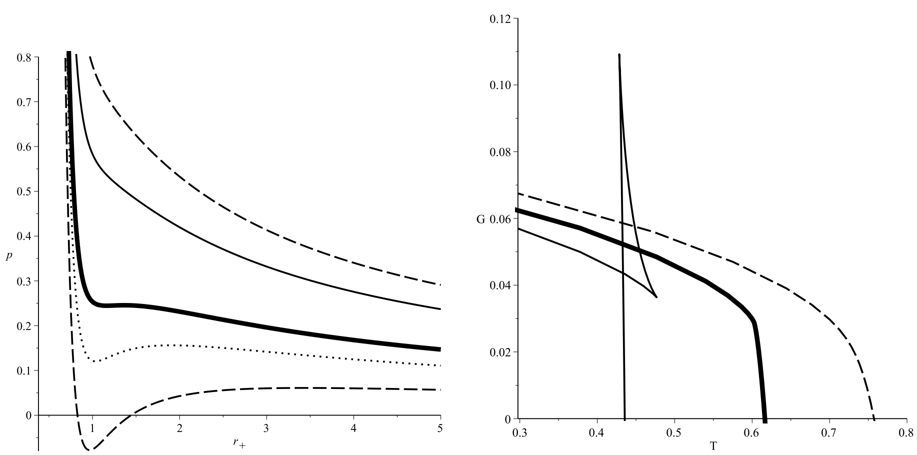

FIG. 5: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 6$, $q = 1$ and $\omega = 3$.
$P - r_+$ diagram, from up to bottom $T = 1.8T_c$, $T = 1.5T_c$, $T = T_c$, $T = 0.8T_c$ and $T = 0.5T_c$ respectively.
$G - T$ diagram, from up to bottom $P = 1.5P_c$, $P = P_c$ and $P = 0.5P_c$, respectively.

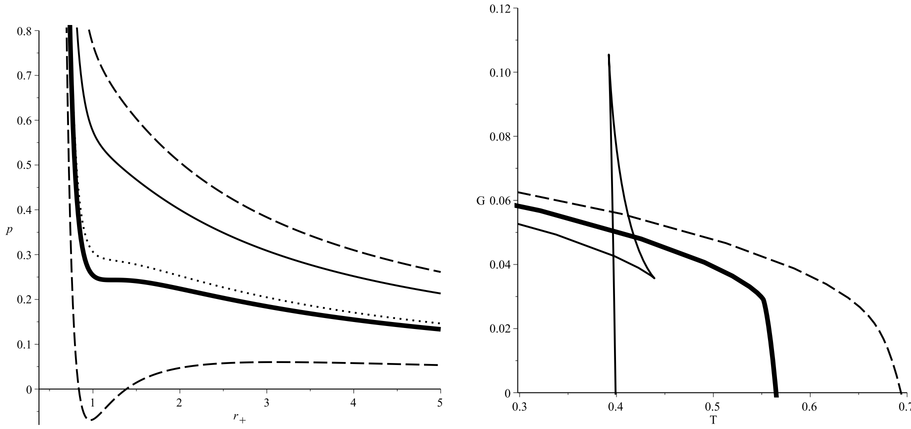

FIG. 6: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 6$, $q = 1$ and $\omega = 5$.
$P - r_+$ diagram, from up to bottom $T = 1.8T_c$, $T = 1.5T_c$, $T = T_c$, $T = 0.8T_c$ and $T = 0.5T_c$ respectively.
$G - T$ diagram, from up to bottom $P = 1.5P_c$, $P = P_c$ and $P = 0.5P_c$, respectively.

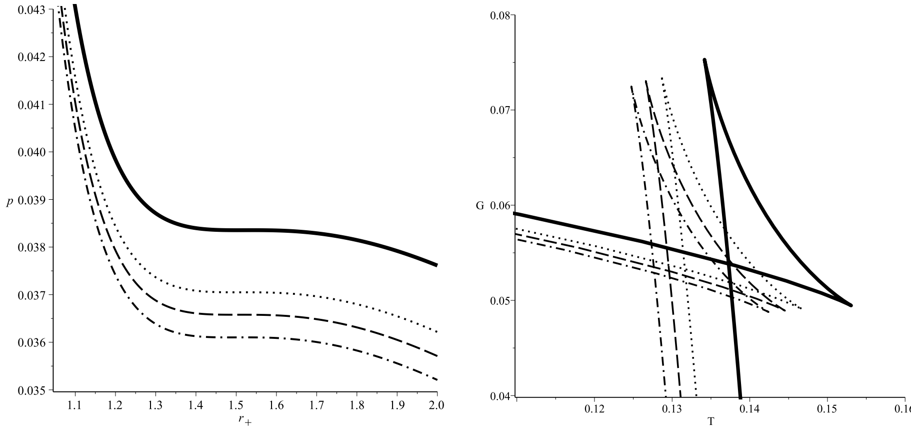

FIG. 7: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 4$, $q = 1$.
$P - r_+$ diagram, for $T = T_c$, $\omega = 1$ (solid line), $\omega = 3$ (dotted line), $\omega = 5$ (dashed line) and $\omega = 10$ (dasheddotted line).
$G - T$ diagram, for $P = 0.5P_c$, $\omega = 1$ (solid line), $\omega = 3$ (dotted line), $\omega = 5$ (dashed line) and $\omega = 10$ (dasheddotted line).

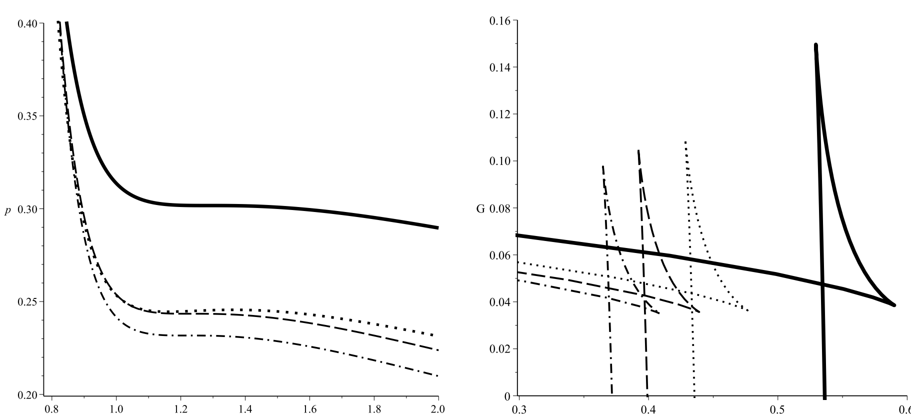

FIG. 8: $P - r_+$ (left), $G - T$ (right) diagrams for $c = 1$, $n = 6$, $q = 1$.
$P - r_+$ diagram, for $T = T_c$, $\omega = 1$ (solid line), $\omega = 3$ (dotted line), $\omega = 5$ (dashed line) and $\omega = 10$ (dasheddotted line).
$G - T$ diagram, for $P = 0.5P_c$, $\omega = 1$ (solid line), $\omega = 3$ (dotted line), $\omega = 5$ (dashed line) and $\omega = 10$ (dasheddotted line).

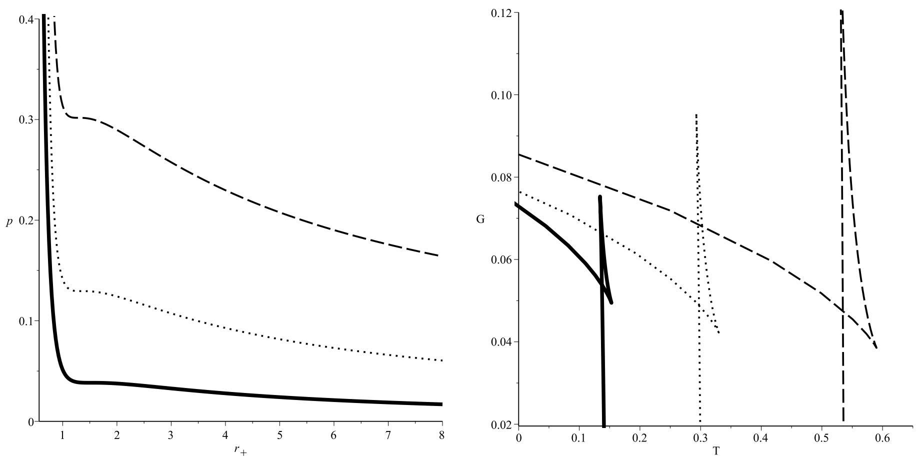

FIG. 9: $P-r_+$ (left), $G-T$ (right) diagrams for $c=1$, $\omega=1$, $q=1$.
$P-r_+$ diagram, for $T=T_c$, $n=4$ (solid line), $n=5$ (dotted line) and $n=6$ (dashed line).
$G-T$ diagram, for $P=0.5P_c$, $n=4$ (solid line), $n=5$ (dotted line) and $n=6$ (dashed line).

Using Fig. 9, we can also discuss the effect of dimensionality on critical point and behavior of system. It can be seen from the obtained $G-T$ diagrams for different dimensions that as dimensionality increases, the temperature of critical point increases too. According to $P-r_+$ diagrams the pressure in which phase transition occurs will increases as dimensionality increases. As a consequence of increment in pressure one can see that the cosmological constant decreases, due to the relation of $P=-\frac{\Lambda}{8\pi}$. Hence the need of having a background with higher value of curvature decreases in higher dimensions. It can be inferred that as dimensionality increases the system needs to absorb more mass in order to have phase transition.

At the end one can see the critical horizon radius decreases as coupling constant increases same as critical temperature and pressure. On the other hand for higher dimensions, we have higher values of critical points. The results arisen from graphs can also be seen directly through the tables I and II.

ω	r_c	T_c	P_c
1.0000	1.5227	0.1869	0.0384
3.0000	1.5099	0.1790	0.0371
5.0000	1.5052	0.1762	0.0366
10.0000	1.5010	0.1736	0.0361

Table I: critical quantities for $q=1$ and $n=4$.

ω	r_c	T_c	P_c
1.0000	1.2859	0.7460	0.3018
3.0000	1.2456	0.5839	0.2627
5.0000	1.2325	0.5517	0.2434
10.0000	1.2210	0.5130	0.2318

Table II: critical quantities for $q=1$ and $n=6$.

VI. CONCLUSIONS

In this paper, we considered the BD theory in the presence of electromagnetic field and studied its phase structure. We extended the phase space by considering cosmological constant as thermodynamical pressure and its conjugate variable as volume and regarded the interpretation of total mass of black hole as the enthalpy of the system.

Studying calculated critical values through two different types of phase diagrams resulted into phase transition taking place in the critical values. Studying $P-r_+$ and $G-T$ diagrams exhibits similar behavior near critical points to their corresponding diagrams in Van der Waals liquid/gas.

The results indicated that for large values of coupling constant the system needs less energy (mass) absorption to have phase transition, due to the fact that as coupling constant increases the critical temperature decreases. On the other hand, studying the effects of dimensionality showed that for higher dimensional black holes, phase transition take places in higher temperature and lower Gibbs free energy.

Finally considering BD theory with various models of nonlinear electrodynamics, it would be interesting to analyze the effects of nonlinearity on extended phase space thermodynamics and $P-V$ criticality of black hole solutions. We left these issues for the forthcoming work.

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